DIVISIBLE POINTS OF COMPACT CONVEX SETS

BY
G. R. WOOD
Department of Mathematics, University of Canterbury, Christchurch, New Zealand

ABSTRACT
By associating with an affine dependence the resultant of a related probability measure, we are able to define the set of *divisible points*, $D(K)$, of a compact convex set K. Some general properties of $D(K)$ are discussed, and its equivalence with a set recently introduced by Reay for convex polytopes demonstrated. For polytopes, $D(K)$ is a continuous image of a projective space. A conjecture concerning $D(K)$ is settled affirmatively for cubes.

1. Introduction and definitions

Within every compact convex set K in a real Hausdorff locally convex space E we will show that there exists, in a natural way, a subset $D(K)$ termed the set of *divisible points* of K. In Section 2, Theorem 1, we demonstrate the part these divisible points play in determining aspects of the structure of K. Section 3 specializes to the case in which K is a convex polytope, where it is shown that $D(K)$ coincides with a set recently introduced by Reay in [14]. Our novel view of this set allows us to present an unexpected result concerning its structure (Theorem 2). Finally (Theorem 3) we confirm a conjecture made by Reay concerning divisible sets, for the special instance of cubes in finite dimensions. A variety of examples, including the Hilbert cube, are discussed in Section 4. The paper serves to relate certain ideas, which have arisen in the study of infinite dimensional convexity, to recently introduced finite dimensional notions.

We follow the notation established by Alfsen in [3]. Throughout, all measures will be in $M(K)$, the dual space of the Banach space of all continuous real-valued functions on K. These are termed the Radon measures on K, and can be identified with all totally finite, regular, signed Borel measures on K, with

Received June 10, 1983 and in revised form October 25, 1984

$\|\mu\| = |\mu|(K)$ for $\mu \in M(K)$. Let $M^+(K)$ be all positive measures in $M(K)$ and $M_1^+(K)$ be all probability measures on K, that is, measures $\mu \in M^+(K)$ with $\mu(K) = 1$.

A measure $\mu \in M(K)$ is said to be a boundary measure if $|\mu|$ is a maximal element of $M^+(K)$ with respect to Choquet's ordering $<$ (if $\mu, \mu' \in M^+(K)$, then $\mu < \mu'$ if $\mu(f) \leqq \mu'(f)$ for all real valued continuous and convex functions f on K). For a metrizable compact convex set K, μ is a boundary measure if and only if it vanishes off the measurable set $\partial_e K$ (the set of extreme points of K). Let $Z(K)$ denote all boundary measures on K, $Z_1^+(K)$ all boundary measures in $M_1^+(K)$ and $Z_x^+(K)$ all boundary probability measures with resultant $x \in K$. Recall that the resultant of $\mu \in M(K)$ [4, Definition 26.2] is x (denoted $r(\mu)$) if and only if $\int_K f(z)d\mu = f(x)$ for all $f \in E^*$, the dual of the real Hausdorff locally convex space E (written $E \in \mathrm{LCS}$). An affine dependence ν on K is a signed boundary measure with $\nu(K) = 0$ and $r(\nu) = 0$, the origin of E. Let $N(K)$ denote the linear space of all such affine dependences on K.

For the benefit of the reader familiar only with finite dimensional convexity, we give an explicit description of $Z_1^+(K)$, $Z_x^+(K)$ and $N(K)$ for $x \in K$, a convex polytope with extreme points $x_1, \dots, x_m$:

$$Z_1^+(K) = \left\{(\alpha_1, \dots, \alpha_m) \in l_1^m : \sum_{i=1}^m \alpha_i = 1 \text{ and } \alpha_i \geqq 0 \text{ for } i = 1, \dots, m\right\},$$

$$Z_x^+(K) = \left\{(\alpha_1, \dots, \alpha_m) \in Z_1^+(K) : \sum_{i=1}^m \alpha_i x_i = x\right\}, \qquad \text{and}$$

$$N(K) = \left\{(\alpha_1, \dots, \alpha_m) \in l_1^m : \sum_{i=1}^m \alpha_i = 0 \quad \text{and} \quad \sum_{i=1}^m \alpha_i x_i = 0\right\},$$

where l_1^m denotes the set of m-tuples of reals, $(\alpha_1, \dots, \alpha_m)$, with

$$\|(\alpha_1, \dots, \alpha_m)\| = \sum_{i=1}^m |\alpha_i|.$$

It is readily shown for any compact convex set K that $Z_x^+(K) = Z_1^+(K) \cap (\mu_x + N(K))$, where $\mu_x \in Z_x^+(K)$.

Finally, $\operatorname{lin} X$ and $\operatorname{aff} X$ denote the linear and affine spans of $X \subseteq E$, while $\dim M$ denotes the dimension of a linear or affine manifold M of E.

2. Divisible points and their role

It is well known that every boundary probability measure on a compact convex set K is associated with an element of K: with each $\mu \in Z_1^+(K)$ we associate the weak μ-integral, $\int_K x d\mu$, the *barycenter* or *resultant* $r(\mu)$ of μ [3, p. 10, Proposition I.2.1]. That this mapping from $Z_1^+(K)$ to K is onto is the content of the Choquet–Bishop–de Leeuw theorem [3, p. 36, Theorem I.4.8].

In an equally natural fashion we can associate an element of K with each one-dimensional subspace of $N(K)$, in the following way. Take a non-zero affine dependence, ν, and consider the Jordan decomposition $\nu = \nu^+ - \nu^-$. Since $\nu(K) = 0$ and $\nu \neq 0$ we have $\nu^+(K) = \nu^-(K) \neq 0$, so $\nu^+/\nu^+(K)$ and $\nu^-/\nu^-(K)$ are boundary probability measures with identical resultants. While these calculations were used by Alfsen in the proof of the theorem in [2], the essence of the method is contained in Radon's paper of 1921 [13]. For this reason we shall refer to the common resultant, w say, as the *Radon-resultant* (or *R-resultant*) of ν, and write $\nu = \nu_w$ or $R(\nu) = w$. Since, as noted earlier, Choquet defines the resultant for any measure in $M(K)$ as the weak integral of the identity function on K this avoids possible confusion. Observe that if ν' is a non-zero real multiple of $\nu \in N(K)$, then $R(\nu') = R(\nu)$.

DEFINITION. Let K be compact and convex in $E \in$ LCS. Then $D(K)$, the *divisible set* of K, is given by

$$D(K) = \{R(\nu) : \nu \text{ is a non-zero element of } N(K)\}.$$

Elements of $D(K)$ we term *divisible points* of K.

The following proposition indicates that $D(K)$ is a non-empty subset of $K \setminus \partial_e K$, as long as K is not a simplex.

PROPOSITION 1. (i) $D(K) = \emptyset$ *if and only if K is a simplex.*
(ii) $D(K) \subseteq K \setminus \partial_e K$ *(so for non-empty K, $D(K)$ is always a proper subset of K).*

PROOF. Statement (i) follows since $N(K) = \{0\}$ if and only if K is a simplex. For (ii) suppose $x \in D(K)$. Then there exist distinct elements in $Z_x^+(K)$ so $x \notin \partial_e K$, since $x \in \partial_e K$ if and only if $Z_x^+(K) = \{\varepsilon_x\}$, where ε_x is the point mass at x.

A pair of non-affinely isomorphic polytopes can have identical divisible sets. For example, the cyclic polytope and the stacked polytope in $\mathbf{R}^4$, each with six extreme points, are not affinely isomorphic [6]. However, the affine dependences of each are one-dimensional, so the divisible sets are singletons. Nevertheless, $D(K)$ does play a central role in determining each $Z_x^+(K)$ set. Recall that $Z_x^+(K) = Z_1^+(K) \cap (\mu_x + N(K))$, where $\mu_x \in Z_x^+(K)$, so we always have $\operatorname{aff} Z_x^+(K) \subseteq \mu_x + N(K)$. The manner in which $D(K)$ produces the affine submanifold of $\mu_x + N(K)$ equal to $\operatorname{aff} Z_x^+(K)$ is the content of Theorem 1.

We pause to recall one more concept before this is presented: the smallest face of K containing $x \in K$ is denoted by face(x) and equals

$$\{w \in K : \text{there exists an } \varepsilon > 0 \text{ such that } x + \varepsilon(x - w) \in K\}.$$

See [3, p. 121].

THEOREM 1. *Let K be a compact convex subset of $E \in$ LCS, $x \in K$ and $\mu_x \in Z_x^+(K)$. If*

$$A_x = \{\nu_w : w \in D(K) \text{ and } w \in \text{face}(x)\},$$

then

$$\text{aff}\, Z_x^+(K) = \mu_x + \text{lin}\, A_x.$$

PROOF. We will verify the following two statements:
(i) if $A_x \neq \emptyset$, then $\text{lin}(Z_x^+(K) - Z_x^+(K)) = \text{lin}\, A_x$, and
(ii) if $A_x = \emptyset$, then $Z_x^+(K)$ is a singleton.
Since for any subset S of E and $s \in S$ we have $\text{aff}\, S = s + \text{lin}(S - S)$, the theorem will follow.

In order to show (i), suppose $A_x \neq \emptyset$ and take $\nu (= \nu_w)$ in A_x. Let $\lambda^+ = \nu^+/\nu^+(K)$ and $\lambda^- = \nu^-/\nu^-(K)$ in the usual way. Since $w \in \text{face}(x)$ we can find a $y \in K$ such that $x = \alpha w + (1-\alpha)y$ with $\alpha \in (0,1]$. Choose $\tau \in Z_y^+(K)$. Then $\mu_1 = \alpha\lambda^+ + (1-\alpha)\tau$ and $\mu_2 = \alpha\lambda^- + (1-\alpha)\tau$ are both boundary probability measures with barycenter x. Also

$$\nu = \nu^+(K)(\lambda^+ - \lambda^-) = \frac{\nu^+(K)}{\alpha}(\mu_1 - \mu_2)$$

so ν lies in $\text{lin}(Z_x^+(K) - Z_x^+(K))$ and $\text{lin}\, A_x \subseteq \text{lin}(Z_x^+(K) - Z_x^+(K))$.

For the reverse inclusion, take distinct μ_1 and μ_2 in $Z_x^+(K)$, possible via the previous argument. Consider the affine dependence $\nu = \mu_1 - \mu_2$ and let w be its R-resultant. In order to show that $\text{lin}(Z_x^+(K) - Z_x^+(K)) \subseteq \text{lin}\, A_x$ it suffices to show that $w \in \text{face}(x)$. Let $A \cup B$ be the Hahn decomposition of K with respect to ν and $\nu^+ - \nu^-$ be the associated Jordan decomposition of ν. As before, let $\lambda^+ = \nu^+/\nu^+(K)$ and $\lambda^- = \nu^-/\nu^-(K)$.

With F an arbitrary measurable subset of K, certainly $\nu^+(F) \geqq 0$, so $\mu_1(A \cap F) \geqq \mu_2(A \cap F)$. Now suppose F is such that $\mu_1(A \cap F) > 0$. Then $\mu_1(F) \geqq \mu_1(A \cap F) > 0$, so

$$\begin{aligned}\frac{\lambda^+(F)}{\mu_1(F)} &= \frac{\nu^+(F)}{\mu_1(F)\nu^+(K)} \\ &= \left[\frac{\mu_1(A\cap F)-\mu_2(A\cap F)}{\mu_1(F)}\right]\frac{1}{\nu^+(K)} \\ &\leqq \left[\frac{\mu_1(A\cap F)-\mu_2(A\cap F)}{\mu_1(A\cap F)}\right]\frac{1}{\nu^+(K)} \\ &= \left[1-\frac{\mu_2(A\cap F)}{\mu_1(A\cap F)}\right]\frac{1}{\nu^+(K)} \\ &\leqq \frac{1}{\nu^+(K)} < \infty.\end{aligned}$$

Thus for all such F there exists an $\alpha > 1$ for which

$$\frac{\lambda^+(F)}{\mu_1(F)} \leqq \frac{\alpha}{\alpha - 1}.$$

All remaining measurable F are such that $\mu_1(A \cap F) = 0$. In this event $\mu_2(A \cap F) = 0$ so

$$\lambda^+(F) = \frac{\nu^+(F)}{\nu^+(K)} = \frac{1}{\nu^+(K)}[\mu_1(A\cap F) - \mu_2(A \cap F)] = 0,$$

whence we certainly have

$$\lambda^+(F) \leqq \left(\frac{\alpha}{\alpha - 1}\right)\mu_1(F).$$

We conclude that $(1-\alpha)\lambda^+(F) + \alpha\mu_1(F) \geqq 0$ for all measurable sets F in K. Thus $\tau = (1-\alpha)\lambda^+ + \alpha\mu_1$ is a boundary probability measure on K with barycenter $(1-\alpha)w + \alpha x$, so $w \in \text{face}(x)$. Since $\mu_1 - \mu_2 = \nu$, $\text{lin}(Z_x^+(K) - Z_x^+(K)) \subseteq \text{lin}\, A_x$.

For (ii), assume that there exist distinct μ_1 and μ_2 in $Z_x^+(K)$. The proof in (i) that $\nu = \mu_1 - \mu_2$ with R-resultant w is such that $w \in \text{face}(x)$ ensures $A_x \neq \emptyset$.

An illustration of the theorem is given after Example 2 in Section 4.

We have already noted that every element of $Z_1^+(K)$ has resultant in K. We conclude this section with a characterization of those elements in $Z_1^+(K)$ whose resultants lie in $D(K)$. Recall that μ and λ in $Z_1^+(K)$ are mutually singular (written $\mu^\perp\lambda$) if there exists a partition $A \cup B$ of K, such that for each measurable set F in K, $A \cap F$ and $B \cap F$ are measurable and $\mu(A \cap F) = \lambda(B \cap F) = 0$. The one-to-one correspondence between one-dimensional subspaces of $N(K)$ and pairs of mutually singular boundary probability measures with the same resultant (in which $(\nu^+/\nu^+(K), \nu^-/\nu^-(K))$ is associated with lin ν, for each non-zero $\nu \in N(K)$) straightforwardly gives the following.

PROPOSITION 2. *Let $P(K)$ be the set of $\mu \in Z_1^+(K)$ such that there exists a $\lambda \in Z_1^+(K)$ with $\mu^\perp\lambda$ and $r(\mu) = r(\lambda)$. Then*

$$D(K) = \{r(\mu): \mu \in P(K)\}.$$

REMARKS. (i) The Hahn decomposition $A \cup B$ of K, associated with such a pair of mutually singular boundary probability measures with the same resultant, can be regarded as a generalisation of the longstanding finite-dimensional notion of a Radon partition [6], since $\overline{\mathrm{co}}\, A \cap \overline{\mathrm{co}}\, B \neq \emptyset$. We explore this relationship further in the next section.

(ii) A means of determining all such Hahn decompositions for certain α-polytopes is given in [9]. The tool used is the Gale transform of the α-polytope.

3. Finite dimensional results

In [14, p. 241] Reay introduced the 2-*divisible* points of a convex polytope K in $\mathbf{R}^n$. These are the points $x \in K$ for which there exists a partition $S \cup T$ of $\partial_e K$ such that $x \in \mathrm{co}\, S \cap \mathrm{co}\, T$. Reay calls this set $D_2(\partial_e K)$. Radon's theorem [13] then asserts that for K in $\mathbf{R}^n$ with $n+2$ extreme points, $D_2(\partial_e K) \neq \emptyset$.

PROPOSITION 3. *Let K be a convex polytope in $\mathbf{R}^n$. Then $D_2(\partial_e K) = D(K)$.*

PROOF. Take $x \in D_2(\partial_e K)$. Then $x \in \mathrm{co}\, S \cap \mathrm{co}\, T$ for some partition $S \cup T$ of $\partial_e K$. Since $x \in \mathrm{co}\, S$, $x = r(\mu)$ for some probability measure μ such that $\mu(K \backslash S) = 0$. Any such measure vanishing off $\partial_e K$ is a boundary measure [3, p. 35, 4.11] so $\mu \in Z_x^+(K)$. Similarly, there exists a $\lambda \in Z_x^+(K)$ for which $\lambda(K \setminus T) = 0$. Certainly $\mu^\perp\lambda$ so $D_2(\partial_e K) \in D(K)$, using the characterization given in Proposition 2.

Now take $x \in D(K)$. From Proposition 2 we know there exists a mutually singular pair of probability measures on $\partial_e K$, μ and λ, each with barycenter x. Here we are again using the fact that boundary probability measures vanish off $\partial_e K$. Thus there is a partition $S \cup T$ of $\partial_e K$ with $\mu(T) = \lambda(S) = 0$. Since $r(\mu) = x$, we have $x \in \mathrm{co}\, S$, and similarly $x \in \mathrm{co}\, T$.

Knowing now that $D_2(\partial_e K)$ is the locus of R-resultants of affine dependences, we are able to present a theorem which indicates that this set has a surprising structure. In the sequel, we revert to the notation $D(K)$ for this set.

Let $\mathbf{P}^m$ be projective m-space. That is, $\mathbf{P}^m$ is the unit sphere in $\mathbf{R}^{m+1}$, S^m, with antipodal points identified, equipped with the quotient topology. Note that for every convex polytope K in $\mathbf{R}^n$, $\dim N(K) < \infty$. See for example [1, p. 100].

THEOREM 2. *Let K be a convex polytope in $\mathbf{R}^n$ and suppose $\dim N(K) = d$. Then $D(K)$ is a continuous image of $\mathbf{P}^{d-1}$.*

PROOF. It is readily checked that the relation ρ on $N(K)\backslash\{0\}$, given by $(\nu, \nu') \in \rho$ if and only if $\nu = \alpha\nu'$ for some $\alpha \in \mathbf{R}$, is an equivalence relation. Define a map $\bar{R}$ from $(N(K)\backslash\{0\})/\rho$ to K by $\bar{R}([\nu]) = R(\nu)$, where $[\nu]$ denotes the equivalence class of ν. This map is well-defined since if $\nu = \alpha\nu'$ and $\alpha \in \mathbf{R}$, then $R(\nu) = R(\nu')$.

Suppose $|\partial_e K| = m$. Then $Z(K)$ is isomorphic to l_1^m, the space of m-tuples of real numbers equipped with the 1-norm. Let S^{m-1} be the unit sphere in l_1^m. Now $D(K) = \bar{R}((N(K) \cap S^{m-1})/\rho)$ and $(N(K) \cap S^{m-1})/\rho$ is certainly homeomorphic to $\mathbf{P}^{d-1}$. By [8, p. 95, Theorem 9] it suffices to show R on $N(K) \cap S^{m-1}$ is continuous.

Let ν_i tend to ν, in $N(K) \cap S^{m-1}$, as i tends to infinity. Let $\partial_e K = \{x_1, \ldots, x_m\}$, $\nu_i = (a_{i1}, \ldots, a_{im})$, $\nu = (a_1, \ldots, a_m)$ and suppose, without loss of generality, that $a_1, \ldots, a_p$ are the strictly positive components of ν while $a_{p+1}, \ldots, a_q$ are the zero components. Provided a_j is non-zero, all a_{ij} have the same sign as a_j, for sufficiently large values of i. For $j = p+1, \ldots, q$ let

$$a_{ij}^+ = \begin{cases} a_{ij} & \text{if } a_{ij} \geqq 0, \\ 0 & \text{if } a_{ij} < 0, \end{cases}$$

so

$$\nu_i^+ = (a_{i1}, \ldots, a_{ip}, a_{i(p+1)}^+, \ldots, a_{iq}^+, 0, \ldots, 0).$$

Hence

$$R(\nu_i) = \left(\sum_{j=1}^{p} a_{ij}x_i + \sum_{j=p+1}^{q} a_{ij}^+ x_j\right) \Big/ \left(\sum_{j=1}^{p} a_{ij} + \sum_{j=p+1}^{q} a_{ij}^+\right)$$

which converges to $(\sum_{j=1}^{p} a_j x_j)/\sum_{j=1}^{p} a_j = R(\nu)$ as $i \to \infty$. This follows since $a_{ij} \to a_j$ for $j = 1, \ldots, p$ and $a_{ij}^+ \to 0$ for $j = p+1, \ldots, q$, as $i \to \infty$.

We highlight the following consequences of Theorem 2.

COROLLARY 1. *Let K be a convex polytope in $\mathbf{R}^n$. Then*

(i) *$D(K)$ is closed and connected, and*

(ii) *if $|\partial_e K| = m$, $D(K)$ is a continuous image of $\mathbf{P}^{m-n-2}$.*

PROOF. Immediately we have (i), while (ii) follows since $N(K)$ has dimension $m-(n+1)$ for such a polytope.

In Problem 5 of [15] conditions are sought which ensure $D(K)$ is convex, and it is conjectured that for convex polytopes in $\mathbf{R}^n$, $\operatorname{co} D(K) = C(K)$, where $C(K)$ (termed the 2-core in [15]) is defined as follows:

$$C(K) = \bigcap \{\operatorname{co} T : T \subseteq \partial_e K \text{ and } |(\partial_e K)\setminus T| = 1\}.$$

The question of the dimension of the affine span of $C(K)$ is also raised.

In the next theorem we show that for cubes of any finite dimension, $D(K)$ is convex and equals $C(K)$. It will also follow that $\operatorname{aff} C(K)$ has dimension n, for $n \geqq 3$.

For convenience, we define a standard cube in $\mathbf{R}^n$,

$$K = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : \max\{|x_1|, \dots, |x_n|\} \leqq 1\},$$

the closed unit ball using the supremum norm. General cubes are treated in the corollary. It can be verified that

$$\partial_e K = V = \{x \in \mathbf{R}^n : x_i = \pm 1 \text{ for each } i\}.$$

THEOREM 3. *Let K be the standard cube in $\mathbf{R}^n$. then*

$$D(K) = C(K) = K \cap \left\{ x \in \mathbf{R}^n : \sum_{i=1}^n |x_i| \leqq n-2 \right\}.$$

PROOF. If $n = 1$ all three sets are empty, while if $n = 2$ all are the origin of $\mathbf{R}^2$. We assume $n \geqq 3$ in the remainder, and present the proof in three steps:

(1) $C(K) = K \cap \{x \in \mathbf{R}^n : \Sigma_{i=1}^n |x_i| \leqq n-2\}$,

(2) $\partial_e C(K) = \{x \in \mathbf{R}^n : |\{i : x_i = 0\}| = 2 \text{ and } |\{i : x_i = \pm 1\}| = n-2\}$,

(3) $D(K) = C(K)$.

PROOF OF (1). For each $v \in V$ and $x \in \mathbf{R}^n$ define $f_v(x) = \langle x, v\rangle$, the usual inner product. For brevity we write $V\setminus\{v\}$ as $V\setminus v$.

We claim $\operatorname{co}(V\setminus v) = K \cap f_v^{-1}(-\infty, n-2]$, for each $v \in V$. Now $f_v(v_i) \leqq n-2$ for each $v_i \in V\setminus v$, since v_i and v have at least one component with opposite sign, while $f_v(v) = n$, so $V \cap f_v^{-1}(-\infty, n-2] = V\setminus v$.

Thus

$$\begin{aligned} K \cap f_v^{-1}(-\infty, n-2] &= \operatorname{co}[(V \cap f_v^{-1}(-\infty, n-2)) \cup (K \cap f_v^{-1}(\{n-2\}))] \\ &= \operatorname{co}[(V \cap f_v^{-1}(-\infty, n-2)) \cup (V \cap f_v^{-1}(\{n-2\}))], \end{aligned}$$

since $K \cap f_v^{-1}(\{n-2\}) = \operatorname{co}\{v' \in V : \langle v', v\rangle = n-2\}$. Hence

$$K \cap f_v^{-1}(-\infty, n-2] = \mathrm{co}(V \cap f_v^{-1}(-\infty, n-2])$$
$$= \mathrm{co}(V \backslash v).$$

Finally,

$$C(K) = \bigcap_{v \in V} \mathrm{co}(V \backslash v)$$
$$= K \cap \bigcap_{v \in V} f_v^{-1}(-\infty, n-2]$$
$$= K \cap \{x \in \mathbf{R}^n : |x_1| + \cdots + |x_n| \leqq n-2\}.$$

Proof of (2). Let F denote the right hand set in the equality to be shown, and take x in F. Suppose $\alpha y + (1-\alpha)z = x$, where $y, z \in C(K)$ and $\alpha \in (0,1)$. Then if $x_i = \pm 1$, since $\alpha y_i + (1-\alpha)z_i = x_i$ and $|y_i|, |z_i| \leqq 1$, we must have $y_i = z_i = x_i$. This occurs for $n-2$ co-ordinates, so the remaining co-ordinates of x, y and z must be zero. Thus $x = y = z$ and $F \subset \partial_e C(K)$.

Suppose now that $x \in C(K) \backslash F$. By considering two cases we show it cannot be extreme.

(i) Case $\Sigma_{i=1}^n |x_i| = n-2$: Without loss of generality we may assume $0 < |x_1|, |x_2| < 1$. If not, then either each component is 0, $+1$ or -1, whence $x \in F$, or just one component differs from these values, whence $\Sigma |x_i| \neq n-2$. Let

$$\varepsilon = \min_{i=1,2} \{|x_i|, |x_i - 1|, |x_i + 1|\}, \qquad x_i^+ = \begin{cases} x_i + \varepsilon & \text{if } x_i > 0 \\ x_i - \varepsilon & \text{if } x_i < 0 \end{cases}$$

and

$$x_i^- = \begin{cases} x_i - \varepsilon & \text{if } x_i > 0 \\ x_i + \varepsilon & \text{if } x_i < 0, \end{cases} \qquad \text{for } i = 1,2.$$

Then since $\varepsilon > 0$, $(x_1^+, x_2^-, x_3, \ldots, x_n)$ and $(x_1^-, x_2^+, x_3, \ldots, x_n)$ are distinct points in $C(K)$ with mid-point x.

(ii) Case $\Sigma_{i=1}^n |x_i| < n-2$: Here, without loss of generality, we may assume $0 \leqq |x_1| < 1$. If not, then $\Sigma_{i=1}^n |x_i| = n$, a contradiction. Choose ε such that $0 < \varepsilon < \min\{(n-2) - \Sigma_{i=1}^n |x_i|, |x_1 - 1|, |x_1 + 1|\}$. Then $(x_1 + \varepsilon, x_2, \ldots, x_n)$ and $(x_1 - \varepsilon, x_2, \ldots, x_n)$ are distinct points in $C(K)$ with x as their mid-point.

Proof of (3). It can be shown that $C(K)$ is the intersection of all closed half-spaces which contain all or all but one point of V. Thus if $x \notin C(K)$, x lies in a closed half space containing at most one point of V. Hence $x \notin D(K)$ so $D(K) \subseteq C(K)$.

In order to show the reverse inclusion we present a partition $S \cup T$ of $\partial_e K$ such that $\partial_e C(K)$, and hence $C(K)$, is a subset of both co S and co T. It follows that $C(K) \subseteq D(K)$ and the theorem is proved. Let

$$S = \{x \in \partial_e K : |\{i : x_i = 1\}| \text{ is even}\}$$

and

$$T = \{x \in \partial_e K : |\{i : x_i = 1\}| \text{ is odd}\} = (\partial_e K) \backslash S.$$

Now take $x \in \partial_e C(K)$, and suppose without loss of generality that $x = (0, 0, x_3, \ldots, x_n)$, with $x_i = \pm 1$ for $i = 3, \ldots, n$. Then $(1, 1, x_3, \ldots, x_n)$ and $(-1, -1, x_3, \ldots, x_n)$ both lie in the same set in the partition and have x as their mid-point. On the other hand, both $(1, -1, x_3, \ldots, x_n)$ and $(-1, 1, x_3, \ldots, x_n)$ lie in the opposite set of the partition, again with x as their mid-point.

Note that since $D(K)$ and $C(K)$ are invariant under affine isomorphism, we have the following corollary.

COROLLARY 2. *Let K' be affinely isomorphic to the standard cube, K. Then* $D(K') = C(K')$.

4. Examples

EXAMPLE 1. In [14, Lemma 6] Reay has the following result. Let K be a convex polytope in $\mathbf{R}^2$ with $m \geqq 6$ extreme points. Then $D(K) = C(K)$.

EXAMPLE 2. Let K be a triangular prism, as shown in Fig. 1, with a, b and c the intersections of the diagonals of the three quadrilateral faces of K. It is readily checked that $D(K) = \text{co}\{a, b\} \cup \text{co}\{b, c\} \cup \text{co}\{c, a\}$, certainly homeomorphic to one-dimensional projective space, with $\nu_a = (1, -1, 1, -1, 0, 0)$, $\nu_b = (0, 0, 1, -1, 1, -1)$ and $\nu_c = (1, -1, 0, 0, 1, -1)$. If $w \in D(K)$ is a convex combination of a and b, say, then ν_w is the corresponding convex combination of ν_a and ν_b.

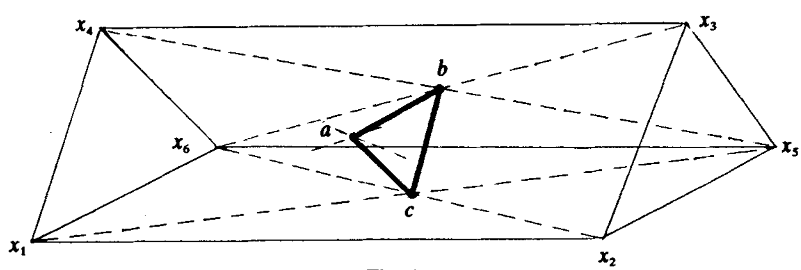

Fig. 1.

This example can be used to illustrate Theorem 1. If x lies on a triangular face, or any edge, no element of $D(K)$ lies in the face generated by x, so $Z_x^+(K)$ is a singleton. For x in the relative interior of a quadrilateral face, one of a, b or c lies in face(x), so $Z_x^+(K)$ will have affine span a translate of $\operatorname{lin}\{\nu_a\}$, $\operatorname{lin}\{\nu_b\}$ or $\operatorname{lin}\{\nu_c\}$ respectively. Finally, if x is in the interior of the prism, all points in $D(K)$ are in face(x), so aff $Z_x^+(K)$ will be a translate of $N(K)$ itself.

EXAMPLE 3. Take $n \geqq 2$, and let K be the closed unit ball in $\mathbf{R}^n$, with the usual topology. Then $D(K)$ is the open unit ball. Since $D(K)$ is open, this demonstrates that Corollary 1(i) does not hold for arbitrary compact convex sets in $\mathbf{R}^n$.

EXAMPLE 4. We present here an example of an infinite dimensional compact convex set K in l_1, the Banach space of all absolutely summable real sequences, together with $D(K)$. Let δ_n be the sequence whose nth term is one and all other terms are zero. Let $x_1 = \delta_1$ and $x_{2n} = \delta_{2n-1}/(2n-1) + \delta_{2n}/2n$, $x_{2n+1} = \delta_{2n}/n + \delta_{2n+1}/(2n+1)$ for $n \geqq 1$. Then $x_i \to x_0 = 0$ in l_1 as $i \to \infty$. Put $X = \{x_i : i = 0, 1, \ldots\}$, and $K = \overline{\mathrm{co}}\, X$, compact by [5, Theorem 6, p. 416]. By [11, p. 9], $\partial_e K \subseteq X$. We indicate that each $x \in X$ is an exposed point of K, hence extreme.

Recall that $z \in K$ is exposed if there exists an $f \in l_\infty$, the dual of l_1, such that $f(x) < f(z)$ for all $x \in K \backslash \{z\}$. We say f exposes z in K. Since K lies in the non-negative cone of l_1, x_0 is exposed by $f \in l_\infty$ for which every term is -1. It is straightforward to show that if $z \in X$ has the property that there exists an $f \in l_\infty$ and $\beta \in \mathbf{R}$ with $f(x) \leqq \beta < f(z)$ for each $x \in X \backslash \{z\}$, then f exposes z in K. For x_i, $i \geqq 1$, we can readily find such an f, and conclude that $\partial_e K = X$.

An affine dependence on K is an l_1 sequence (α_n) such that $\sum_{n=0}^{\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n x_n = 0$, where the second series converges in l_1. From the construction of X it follows that $N(K) = \operatorname{lin}\{\nu\}$, where $\nu = (0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{8}, \ldots)$. Since $\nu^+(K) = 1$, the mutually singular boundary probability measures associated with ν are $\nu^+ = (0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, \ldots)$ and $\nu^- = (0, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \ldots)$. Hence $D(K)$ contains the one point,

$$r(\nu^+) = \tfrac{1}{2}\delta_1 + \tfrac{1}{4}(\delta_2 + \tfrac{1}{3}\delta_3) + \tfrac{1}{8}(\tfrac{1}{2}\delta_4 + \tfrac{1}{5}\delta_5) + \cdots$$

$$= \left(\frac{1}{1.2}, \frac{1}{2.2}, \frac{1}{3.4}, \frac{1}{4.4}, \ldots, \frac{1}{(2n-1)2^n}, \frac{1}{(2n)2^n}, \ldots \right).$$

Note that

$$r(\nu^-) = \tfrac{1}{2}(\delta_1 + \tfrac{1}{2}\delta_2) + \tfrac{1}{4}(\tfrac{1}{3}\delta_3 + \tfrac{1}{4}\delta_4) + \tfrac{1}{8}(\tfrac{1}{5}\delta_5 + \tfrac{1}{6}\delta_6) + \cdots$$

$$= r(\nu^+).$$

EXAMPLE 5. Let

$$K = \{(\xi_n) : |\xi_n| \leqq 1/n, \text{ for each } n \in \mathbf{N}\}$$
$$= \prod_{n=1}^{\infty} [-1/n, 1/n],$$

be the Hilbert cube in l_2, where l_2 denotes the Hilbert space of all square summable real sequences. See for example [10, p. 29]. Phelps [12, Proposition 2.8] has shown that any infinite dimensional centrally symmetric compact convex set, such as the Hilbert cube, necessarily has $N(K)$ infinite dimensional. This indicates, loosely speaking, that $D(K)$ will be a large subset of K. It is a straightforward exercise to show that

$$\partial_e K = \{(\xi_n) : \xi_n = \pm 1/n, \text{ for each } n \in \mathbf{N}\},$$

and interesting to note that $\partial_e K$ is homeomorphic to the Cantor set $\Pi_{n=1}^{\infty}\{-1,1\}$, under the natural map.

We culminate this example with (iii) below, where we sandwich $D(K)$ between a pair of subsets of the Hilbert cube. For this we need necessary and sufficient conditions for $\mu \in Z_1^+(K)$ to have resultant $x = (\xi_n)$, given in (ii), which in turn hinges upon the result we now present.

(i) *$\langle x, y \rangle = \int_{\partial_e K} \langle z, y \rangle d\mu(z)$ for each $y \in l_2$ if and only if $\langle x, e_n \rangle = \int_{\partial_e K} \langle z, e_n \rangle d\mu(z)$ for each $n \in \mathbf{N}$* (where $\langle x, y \rangle$ denotes the inner product and (e_n) the standard basis in l_2).

Since the necessity is immediate, take $y = (\eta_n) \in l_2$ and let $f_n(z) = \Sigma_{i=1}^{n} \eta_i \langle z, e_i \rangle$, for each $n \in \mathbf{N}$ and $z \in \partial_e K$. Choose $r = (\rho_i) \in \partial_e K$ such that $\rho_i \eta_i \geqq 0$ for each i. Then

$$f_n(z) = \sum_{i=1}^{n} \eta_i \langle z, e_i \rangle \leqq \sum_{i=1}^{\infty} |\eta_i|/i \leqq \sum_{i=1}^{\infty} \eta_i \langle r, e_i \rangle = \langle r, y \rangle = M < \infty, \text{for each } n \text{ and } z.$$

Now $\mu(K) = 1$, so it follows from Lebesgue's Bounded Convergence Theorem that

$$\lim_n \int_{\partial_e K} f_n(z) d\mu = \int_{\partial_e K} \lim_n f_n(z) d\mu.$$

Hence

$$\langle x, y \rangle = \left\langle x, \sum_{n=1}^{\infty} \eta_n e_n \right\rangle$$
$$= \sum_{n=1}^{\infty} \eta_n \langle x, e_n \rangle$$

$$= \sum_{n=1}^{\infty} \eta_n \int_{\partial_e K} \langle z, e_n \rangle d\mu$$

$$= \int_{\partial_e K} \sum_{n=1}^{\infty} \eta_n \langle z, e_n \rangle d\mu$$

$$= \int_{\partial_e K} \langle z, y \rangle d\mu.$$

Now let $B_n^+ = \{(\eta_i) \in \partial_e K : \eta_n = 1/n\}$ and $B_n^- = \{(\eta_i) \in \partial_e K : \eta_n = -1/n\}$.

(ii) $r(\mu) = x = (\xi_n)$ *if and only if* $\xi_n = (\mu(B_n^+) - \mu(B_n^-))/n$ *for all* $n \in \mathbf{N}$.

Note that

$$r(\mu) = x \Leftrightarrow \langle x, y \rangle = \int_{\partial_e K} \langle z, y \rangle d\mu, \qquad \text{for each } y \in l_2$$

$$\Leftrightarrow \langle x, e_n \rangle = \int_{\partial_e K} \langle z, e_n \rangle d\mu, \qquad \text{for each } n \in \mathbf{N} \quad \text{(by (i))}$$

$$\Leftrightarrow \xi_n = (\mu(B_n^+) - \mu(B_n^-))/n, \qquad \text{for each } n \in \mathbf{N}.$$

Let A be the set of points $(\xi_n) \in K$ such that, for each, there exists a finite set of indices, $n_1, \ldots, n_N$, with $\sum_{i=1}^{N} (1 - n_i |\xi_{n_i}|) \geqq 2$. For example, any $(\xi_n) \in K$ for which $\sum_{n=1}^{\infty} (1 - n|\xi_n|) > 2$ lies in A. Let B be the set of points $(\xi_n) \in K$ such that, for each, $\xi_n = \pm 1/n$ for all but a finite number of components, $n_1, \ldots, n_N$, say, for which $\sum_{i=1}^{N} (1 - n_i |\xi_{n_i}|) < 2$. Note that $\partial_e K \subseteq B$. We now show

(iii) $A \subseteq D(K) \subseteq K \backslash B$.

Suppose $x = (\xi_n) \in A$. Without loss of generality, assume $n_i = i$, $i = 1, \ldots, N$, so $\sum_{n=1}^{N} (1 - n|\xi_n|) \geqq 2$ or $\sum_{n=1}^{N} n|\xi_n| \leqq N - 2$. Thus $(\xi_1, \ldots, \xi_N)$ is divisible in $K_1 = \prod_{n=1}^{N} [-1/n, 1/n] \subseteq \mathbf{R}^N$, by Theorem 3, hence there exist mutually singular Borel probability measures μ_1 and μ_1' on

$$\partial_e K_1 = P = \{(\eta_1, \ldots, \eta_N) \in K_1 : |\eta_n| = \pm 1/n, \text{for } n = 1, \ldots, N\},$$

each with resultant $(\xi_1, \ldots, \xi_N)$. Now let $K_2 = \prod_{n=N+1}^{\infty} [-1/n, 1/n] \subseteq l_2$. Since $(\xi_{N+1}, \xi_{N+2}, \ldots) \in K_2$, by the Choquet–Bishop–de Leeuw theorem we know there is a Borel probability measure μ_2 on $\partial_e K_2 = Q = \{(\xi_n) \in K_2 : \xi_n = \pm 1/n$, for $n > N\}$, with resultant $(\xi_N, \xi_{N+1}, \ldots)$. By [7, Theorems 50.E and 51.E] we know $B(P) \times B(Q) = B(P \times Q)$, where $B(P)$ denotes the σ-algebra of Borel subsets of P. Thus $\mu = \mu_1 \times \mu_2$ and $\mu' = \mu_1' \times \mu_2$ are Borel probability measures on $P \times Q = \partial_e K$. It is straightforward to check, using (ii), that $r(\mu) = r(\mu') = x$ and also that $\mu \perp \mu'$. We conclude that x is a divisible point of $K = K_1 \times K_2$.

Suppose now that $x \in B$. As before we can assume that $n_i = i$, $i = 1, \dots, N$, so $\sum_{n=1}^{N}(1 - n|\xi_n|) < 2$, and also that $\xi_n = 1/n$, say, for $n > N$. If $\mu \in Z_x^+(K)$ we have

$$\xi_n = (\mu(B_n^+) - \mu(B_n^-))/n = (1 - 2\mu(B_n^-))/n = 1/n,$$

or $\mu(B_n^-) = 0$, for $n > N$. Thus $\mu(\bigcup_{n>N} B_n^-) = 0$, or μ is supported by $(\prod_{n=1}^{N}[-1/n, 1/n]) \times (\prod_{n>N}\{1/n\})$, a finite dimensional cube. Since $\sum_{n=1}^{N} n|\xi_n| > N - 2$, $Z_x^+(K)$ must be a singleton, or $x \notin D(K)$.

To illustrate (iii), note that $((n-1)/n^2) \in D(K)$, whereas, for each fixed $N \in \mathbf{N}$, $(\xi_n) \notin D(K)$, where $\xi_n = (N-1)/nN$ for $n \leqq N$ and $\xi_n = 1/n$ for $n > N$.

Concluding Remarks. In Theorem 2 we showed that if K is a convex polytope in $\mathbf{R}^n$ with $m \geqq 2n + 2$ extreme points, so $\dim N(K) = m - (n+1) \geqq n + 1$, then $D(K)$ is the continuous image of a projective space with dimension greater than or equal to n. This suggests the following sharpening of Reay's conjecture [15, p. 155, Problem 5(b)].

Conjecture 1. Let K be a convex polytope in $\mathbf{R}^n$ $(n > 1)$ with $m \geqq 2n + 2$ extreme points. Then $D(K) = C(K)$.

Example 1 and Theorem 3 confirm that for $n = 2$ and 3, and $m = 2n + 2$, then $D(K) = C(K)$.

Using $Z_1^+(K)$, this problem can be translated into an apparently more tractable problem concerning slices through simplexes. Let T^d be the d-simplex, where $d \in \mathbf{N}$.

Conjecture 2. Take $n \in \mathbf{N}$, $m \geqq 2n + 2$ and let M be an $m - (n+1)$ dimensional flat which cuts every facet of T^{m-1} in other than an extreme point. Then M cuts a disjoint pair of faces, each having dimension greater than zero, of T^{m-1}.

A sketch of the case where $n = 1$ and $m = 4$ (a slice through a tetrahedron) illuminates this conjecture. That the validity of the second conjecture implies the validity of the first can be seen as follows. Take $x \in C(K)$. Recall that $Z_x^+(K) = Z_1^+(K) \cap (\mu_x + N(K))$, with $Z_1^+(K)$ an $(m-1)$ dimensional simplex and $\mu_x + N(K)$ an $m - (n+1)$ dimensional flat cutting this simplex. Since $x \in \operatorname{co} T$ for each $T \subseteq \partial_e K$ such that $|(\partial_e K) \setminus T| = 1$, $\mu_x + N(K)$ intersects each facet of $Z_1^+(K)$ in other than an extreme point. If Conjecture 2 holds, $\mu_x + N(K)$ intersects a disjoint pair of faces of dimension greater than zero in $Z_1^+(K)$, so $x \in D(K)$.

Finally, it would be interesting to obtain an explicit description of the divisible points of the Hilbert cube.

ACKNOWLEDGEMENTS

The author wishes to thank Professors T. E. Armstrong, Peter Kleinschmidt and R. R. Phelps for helpful advice during the preparation of this paper.

REFERENCES

1. Erik M. Alfsen, *On the geometry of Choquet simplexes*, Math. Scand. **15** (1964), 97–110.

2. Erik M. Alfsen, *A measure theoretic characterization of Choquet simplexes*, Math. Scand. **17** (1965), 106–112.

3. Erik M. Alfsen, *Compact convex sets and boundary integrals*, Ergebnisse der Math. Vol. 57, Springer-Verlag, Berlin, 1971.

4. G. Choquet, *Lectures on Analysis* (J. Marsden, T. Lance and S. Gelbart, eds.), Benjamin, Inc., New York, 1969.

5. Nelson Dunford and Jacob T. Schwartz, *Linear Operators, Part I*, Interscience, New York, London, Sydney, 1966.

6. Branko Grünbaum, *Convex Polytopes*, Interscience, New York, London, Sydney, 1967.

7. Paul R. Halmos, *Measure Theory*, Van Nostrand Co., Princeton, N.J., 1950.

8. John L. Kelley, *General Topology*, Van Nostrand Co., Princeton, N.J., 1955.

9. P. Kleinschmidt and G. R. Wood, *Gale transforms and closed faces of infinite dimensional polytopes*, Mathematika **31** (1984), 291–304.

10. Gottfried Köthe, *Topological Vector Spaces I* (translated by D. J. H. Garling), Die Grundlehren der mathematischen Wissenschaften, Berlin, Heidelberg, New York, 1969.

11. R. R. Phelps, *Lectures on Choquet's Theorem*, Van Nostrand Math. Studies 7, Princeton, N.J., 1966.

12. R. R. Phelps, *Infinite dimensional compact convex polytopes*, Math. Scand. **24** (1969), 5–26.

13. J. Radon, *Mengen Konvexer Körper, die einen gemeinsamen Punkt enthalten*, Math. Ann. **83** (1921), 113–115.

14. John R. Reay, *Several generalizations of Tverberg's theorem*, Israel J. Math. **34** (1979), 238–244.

15. John R. Reay, *Open problems around Radon's theorem*, in *Proceedings of the Second University of Oklahoma Conference on Convexity and Related Combinatorial Geometry* (David C. Kay and Marilyn Breen, eds.), Marcel Dekker, Inc., New York and Basel, 1982.